The stability of elastico-viscous flow between rotating cylinders. Part 2

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Further consideration is given to the stability of the flow of an idealized elastico-viscous liquid contained in the narrow channel between two rotating coaxial cylinders. The work of Part 1 (Thomas & Walters 1964) is extended to include highly elastic liquids. To facilitate this, use is made of the orthogonal functions used by Reid (1958) in his discussion of the associated Dean-type stability problem. It is shown that the critical Taylor number T_c decreases steadily as the amount of elasticity in the liquid increases, until a transition is reached after which the roots of the determinantal equation which determines the Taylor number T as a function of the wave-number ϵ become complex. It is concluded that the principle of exchange of stabilities may not hold for highly elastic liquids.

1. Introduction

In the first paper under the same title (Thomas & Walters 1964) consideration was given to the stability of the flow of an idealized elastico-viscous liquid contained between rotating coaxial cylinders. The particular elastico-viscous liquid considered in the investigation was that designated liquid B′ by Walters (1964). It was shown that the perturbation equations could be written in the form

$$[1+\alpha x]\,v = \nabla_1^4\chi + 2k\alpha\, dv/dx - 3s\epsilon^2\alpha T\, d\chi/dx, \tag{1}$$

$$\nabla_1^2 v = -T\epsilon^2[\chi - k\nabla_1^2\chi], \tag{2}$$

where

$$\left.\begin{aligned} &\nabla_1^2 = d^2/dx^2-\epsilon^2, \quad d = r_2-r_1, \quad r = r_1+dx,\\ &\alpha = (\Omega_2/\Omega_1)-1, \quad \eta_0 = \int_0^\infty N(\tau)\,d\tau, \quad k = \int_0^\infty \tau N(\tau)\,d\tau\Big/\rho d^2,\\ &\qquad\qquad\qquad\qquad\qquad\qquad s = \eta_0\int_0^\infty \tau^2 N(\tau)\,d\tau\Big/\rho^2 d^4, \end{aligned}\right\} \tag{3}$$

$N(\tau)$ being the distribution function of relaxation times (Walters 1960) and ρ the density. In these equations, Ω_1 and Ω_2 are the angular velocities of the inner and outer cylinders of radii r_1 and r_2, respectively. The boundary conditions to be associated with equations (1) and (2) are

$$v = \chi = d\chi/dx = 0 \quad \text{on} \quad x = 0 \quad \text{and} \quad x = 1. \tag{4}$$

Equations (1) and (2) subject to (4) determine a characteristic-value problem for the Taylor number $T\,(= -2\alpha\Omega_1^2 d^3 r_1 \rho^2/\eta_0^2)$ as a function of the wave-number ϵ.

In Part 1, the discussion was limited to liquids with short memory. In this case, terms in s and k^2 could be neglected and it was then possible to use a simple method of solution introduced by Chandrasekhar (1954). It was shown that the presence of a very small amount of elasticity in the liquid could be a significant destabilizing agent.

In the present paper, we reconsider the problem for larger values of k and s, using the method of solution employed by Reid (1958) in his discussion of the Dean-type viscous stability problem and by Thomas & Walters (1963) in their discussion of the same problem for an elastico-viscous liquid. So far as the authors are aware the method of solution used in the present paper has not been employed hitherto in the Taylor-stability problem.

The details of the solution are omitted since the method of analysis is precisely the same as in the papers mentioned above. The problem resolves itself into finding a solution of a doubly-infinite determinant of the form

$$\begin{vmatrix} (A_{nm} - B_{nm}/\epsilon^2 T) & C_{nm} \\ D_{nm} & (E_{nm} - F_{nm}/\epsilon^2 T) \end{vmatrix} = 0, \tag{5}$$

where $A_{nm}, \ldots, F_{nm}$ are functions of ϵ but are independent of T (cf. Reid 1958; Thomas & Walters 1963). Equation (5) is considered as an equation in the two variables T and ϵ, and the critical Taylor number T_c at which the laminar flow pattern breaks down is determined by calculating the minimum T for varying ϵ.

2. Numerical calculation concerned with approximate solutions of the determinantal equation

The simplest approximation to equation (5) (referred to in the following as the first approximation) is that in which only four terms are retained in the determinant. The first approximation was used in most of the calculations of the present paper, the second approximation (involving a determinant of sixteen terms) being used occasionally to check the accuracy of the first approximation.

Table 1 contains a comparison of the values of T (to four significant figures) for the first and second approximations for a Newtonian viscous liquid ($k = s = 0$). In the case when $\alpha = 0$, the exact value obtained by Pellew & Southwell (1940) is given, and in the cases $\alpha = -0{\cdot}5$ and $\alpha = -1{\cdot}0$ the values given by Chandrasekhar (1961) are included.

Figure 1 contains curves of the critical Taylor number T_c as a function of k (with $s = k^2$)† for four values of α. It will be observed that T_c decreases steadily with increasing k, being very sensitive to changes in k for small values of k. The results given in figure 1 are in the region where the first approximation is within 2 % of the second approximation. Table 2 contains the corresponding values of ϵ_c.

For higher values of k and s and non-zero values of α there is a region where the values of T obtained from the first and second approximations are complex.

† For an elastico-viscous liquid with a discrete relaxation spectrum, it can be shown that $s \geqslant k^2$ with equality when the liquid is characterized by one relaxation time (the Maxwell liquid). We have taken $s = k^2$ for the purpose of illustration.

The case $\alpha = -0{\cdot}5$, $k = 0{\cdot}8$, $s = 0{\cdot}64$ has been considered in some detail using the second approximation. No real values of T were found for any value of ϵ. In the case when $\alpha = -0{\cdot}5$, the transition to unreal roots occurs in a region where the first approximation is still good. It therefore appears likely that the principle of exchange of stabilities does not hold for highly elastic liquids and overstability is possible.

	1st approx.	2nd approx.	Exact value
$\alpha = 0, \quad \epsilon = 3{\cdot}11$	1719	1709	1708
$\alpha = -0{\cdot}5, \ \epsilon = 3{\cdot}11$	2290	2277	2275
$\alpha = -1{\cdot}0, \ \epsilon = 3{\cdot}11$	3414	3393	3390

TABLE 1. Values of T.

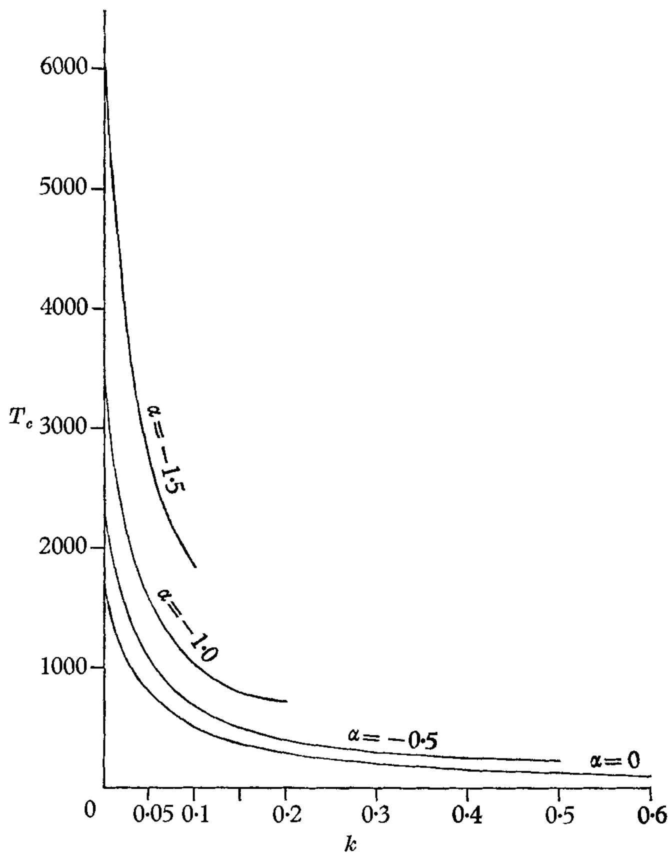

FIGURE 1. Values of T_c against k for various values of α ($s = k^2$).

The theoretical results also indicate that the greater the value of α (in a negative sense) the sooner overstability is likely to occur. This suggests that any given elastico-viscous liquid of the type considered is likely to exhibit overstability provided the relative rotation of the cylinders is high enough.

When the possibility of overstability was first noted, the authors decided to extend the computation associated with their treatment of the corresponding Dean-type stability problem (Thomas & Walters 1963). They found that the critical Taylor number decreased steadily with increasing values of k and s (in agreement with the findings of the present paper) but there was no evidence of overstability even for very large values of k and s.

	ϵ_c			
k	$\alpha = 0$	$\alpha = -0{\cdot}5$	$\alpha = -1{\cdot}0$	$\alpha = -1{\cdot}5$
0·0	3·11	3·11	3·11	3·15
0·05	3·85	3·85	3·85	4·05
0·1	4·15	4·15	4·13	3·85
0·15	4·31	4·28	4·0	
0·2	4·40	4·33	3·6	
0·3	4·48	4·21		
0·4	4·55	3·90		
0·5	4·59	3·48		
0·6	4·61			

TABLE 2.

We are led to the general conclusion that highly elastic liquids can be very unstable and that overstability is possible in the flow of highly elastic liquids between rotating coaxial cylinders.

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